LANGUAGES OF COUNTABLE WORDS

Lorem ipsum (Don't read this! Better listen to the talk) [illegible] Still reading?) ullam corporis suscipit laboriosam, nisi...

Gabriele Puppis

LaBRI / CNRS

based on joint works with

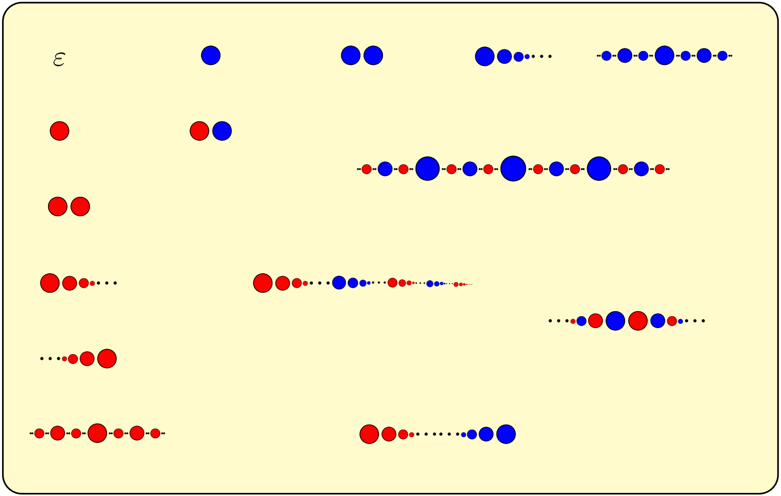
Olivier Carton,
Thomas Colcombet

Given an alphabet $A = \{{\color{red}\bullet}, {\color{blue}\bullet}\}$, let

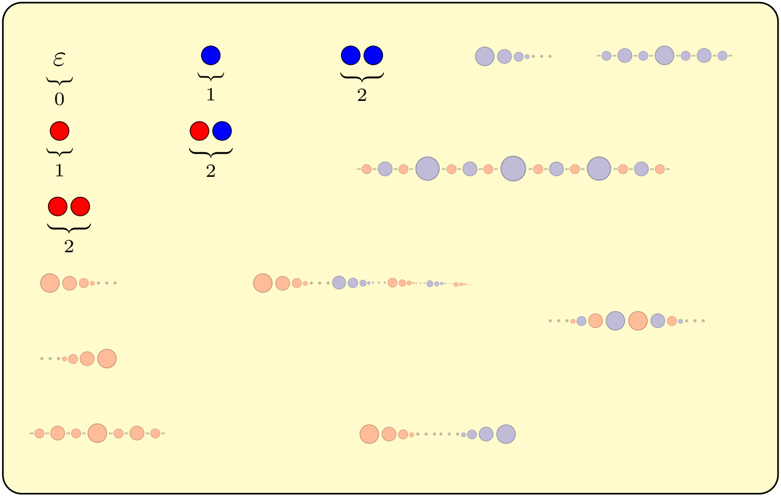
A° = $\Big\{$all **countable words** on $A\Big\}$

Given an alphabet $A = \{\ ,\ \}$, let

$A^\circ =$



Given an alphabet $A = \{$●, ●$\}$, let

$A^\circ =$ 

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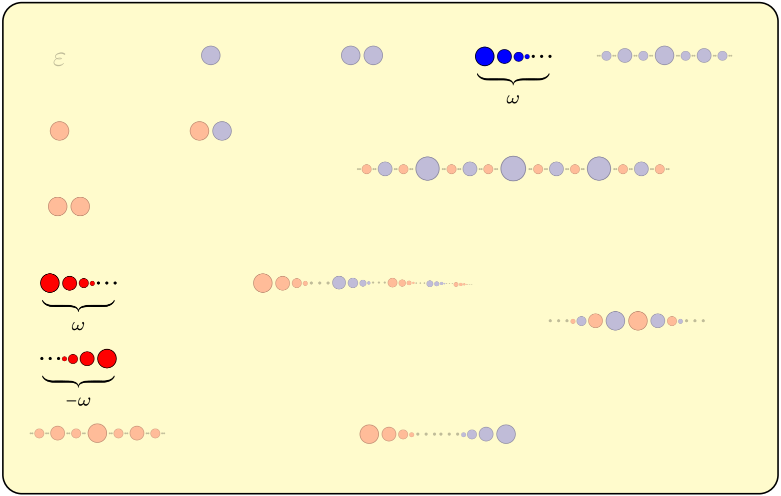

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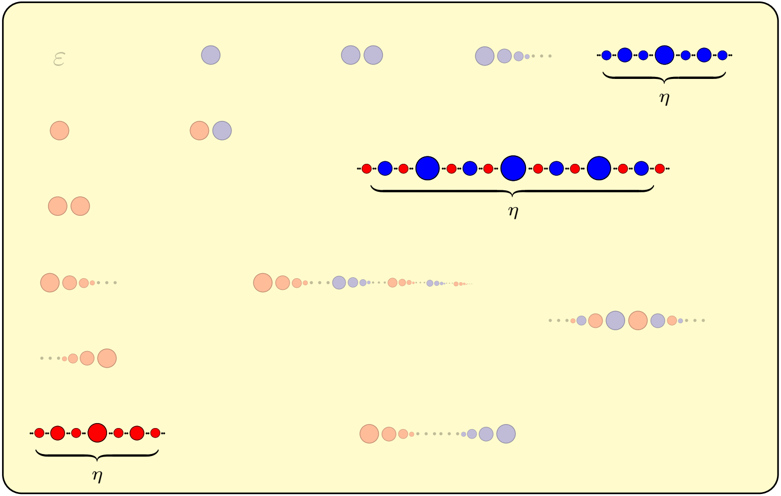

Given an alphabet $A = \{$🔴, 🔵$\}$, let

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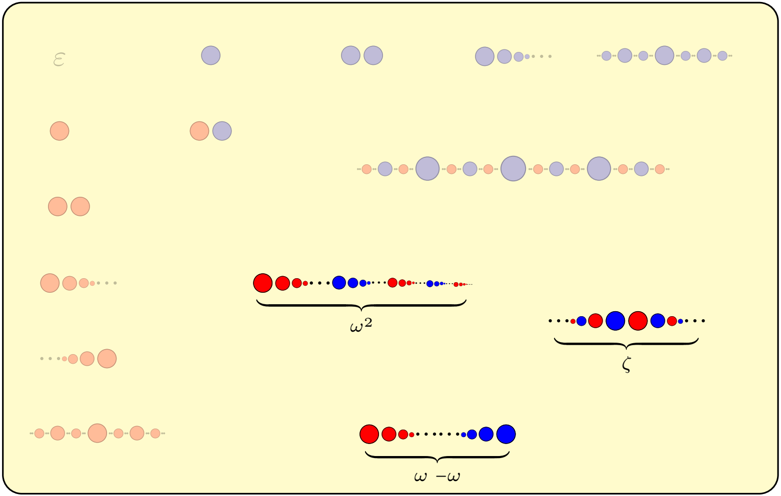

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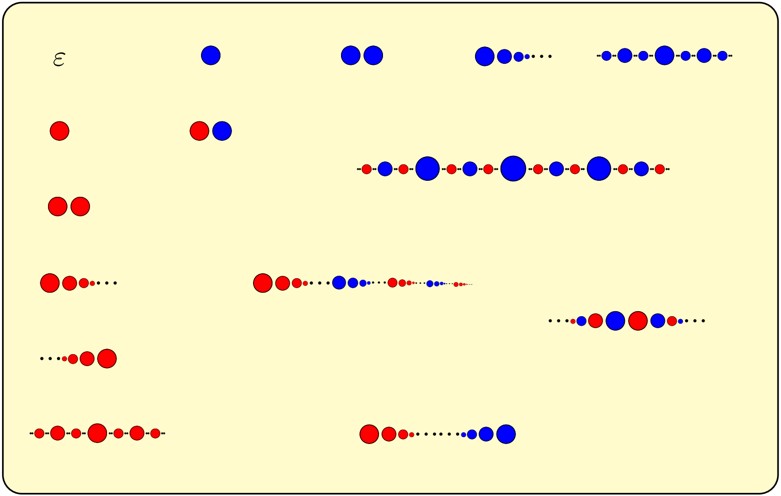
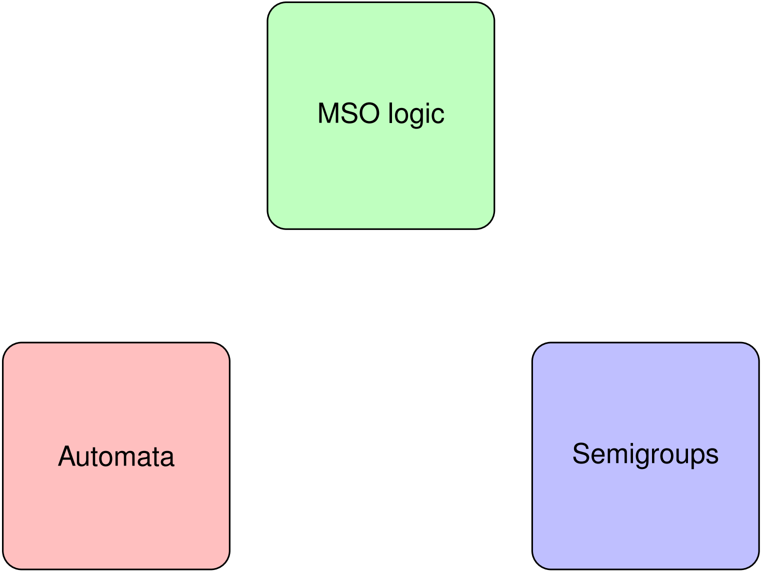


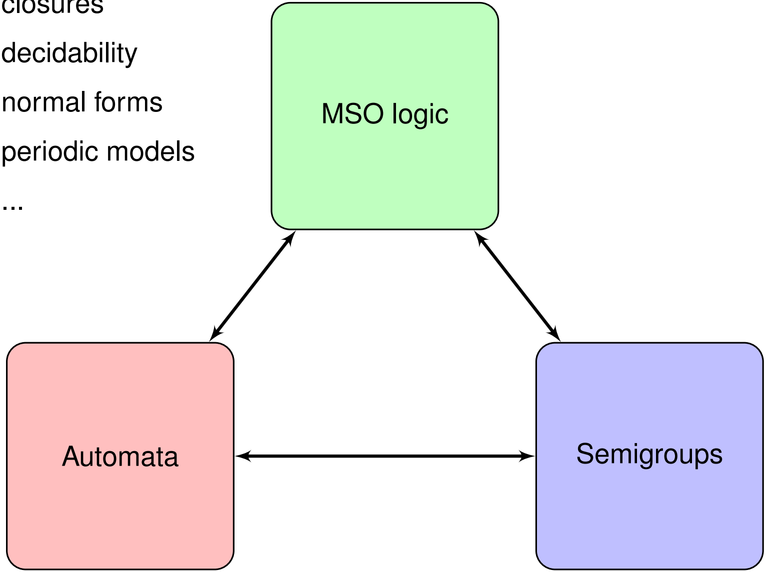
Interest on “**regular**” (= robust & decidable) languages $L \subseteq A^\circ$

Formalisms for classical regular languages

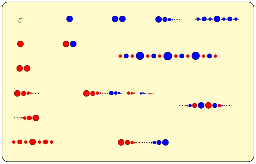


Formalisms for classical regular languages

- ✓ closures
- ✓ decidability
- ✓ normal forms
- ✓ periodic models
- ✓ ...

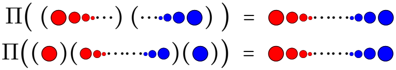


Recognizability of languages via semigroups

$A^\circ =$ 

associative product $\Pi : (A^\circ)^\circ \to A^\circ$

e.g.



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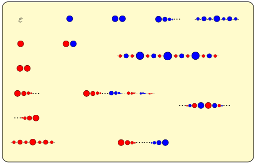
$A^\circ =$ 

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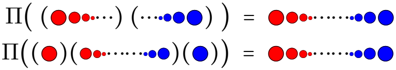
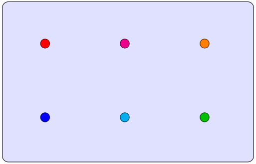
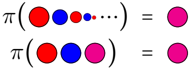
e.g. 

$S =$ 

associative product $\pi : S^\circ \to S$

e.g. 

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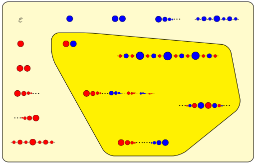
$A^\circ =$ 

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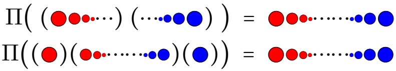
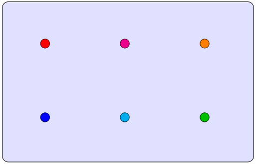
e.g. 

$S =$ 

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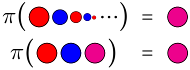
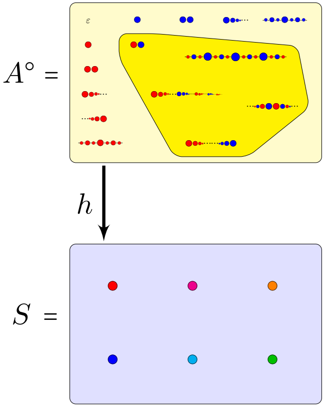
e.g. 

$L \subseteq A^\circ$ **recognized by** (S, π) $\Leftrightarrow$ ...

Recognizability of languages via semigroups



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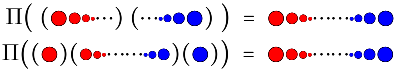

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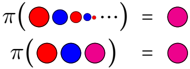
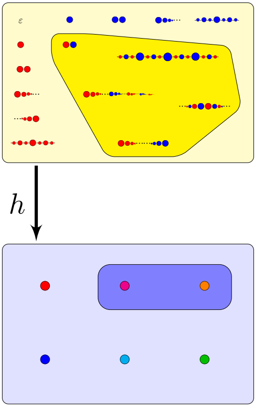


$L \subseteq A^\circ$ **recognized by** (S, π) $\Leftrightarrow$ $\exists\, h : (A^\circ, \Pi) \to (S, \pi)$

...

Recognizability of languages via semigroups

$A^\circ =$



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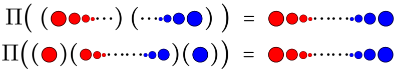

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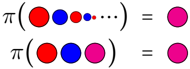

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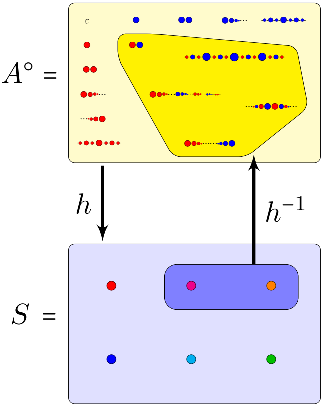

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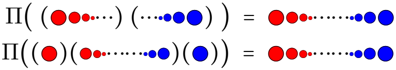

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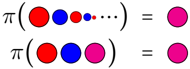

$L \subseteq A^\circ$ **recognized by** (S, π) $\Leftrightarrow$ $\exists\, h : \ (A^\circ, \Pi) \to (S, \pi)$
$\exists\, F \subseteq S$
$L = h^{-1}(F)$

Properties of recognizable languages

Closure under complementations, unions, projections, ...

e.g. if $L = h^{-1}(F)$ then $A^{\circ} \smallsetminus L = h^{-1}(S \smallsetminus F)$

Properties of recognizable languages

- Closure under complementations, unions, projections, ...

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- Capture all languages definable in MSO

  negations → complementations
  disjunctions → unions
  existential quantifications → projections

Properties of recognizable languages

Closure under complementations, unions, projections, ...

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Capture all languages definable in MSO

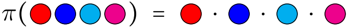
negations $\rightarrow$ complementations
disjunctions $\rightarrow$ unions
existential quantifications $\rightarrow$ projections

Algorithms for emptiness, universality, ... ?

PROBLEM: need to finitely represent countable products!

Algebras as representations of countable products

We use the same approch as in classical semigroups

i.e. 

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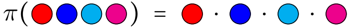
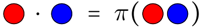
i.e. 

Given a semigroup (S, π), the **induced algebra** consists of

- binary product $\cdot \; : \; S \times S \;\to\; S$



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- $\pm\omega$-powers $\pm\omega \; : \; S \to S$

  $\textcolor{red}{\bullet}^{\omega} = \pi(\textcolor{red}{\bullet}\textcolor{red}{\bullet}\textcolor{red}{\bullet}\textcolor{red}{\bullet}\textcolor{red}{\bullet}\cdots)$

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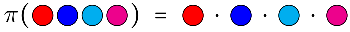

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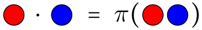

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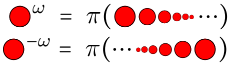

- perfect shuffle $\eta \; : \; \mathscr{P}(S) \to S$

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  $$\{\textcolor{red}{\bullet},\textcolor{blue}{\bullet},\textcolor{green}{\bullet}\}^{\eta} = \pi(\cdots\textcolor{red}{\bullet}\textcolor{blue}{\bullet}\textcolor{green}{\bullet}\textcolor{red}{\bullet}\textcolor{blue}{\bullet}\textcolor{green}{\bullet}\textcolor{red}{\bullet}\textcolor{blue}{\bullet}\textcolor{green}{\bullet}\textcolor{red}{\bullet}\textcolor{blue}{\bullet}\textcolor{green}{\bullet}\textcolor{red}{\bullet}\textcolor{blue}{\bullet}\textcolor{green}{\bullet}\cdots)$$

Equations derived from associativity

e.g. if $\{\textcolor{red}{\bullet},\textcolor{blue}{\bullet},\textcolor{green}{\bullet}\}^{\eta} = \textcolor{magenta}{\bullet}$ then $\textcolor{magenta}{\bullet}\cdot\textcolor{blue}{\bullet}\cdot\textcolor{magenta}{\bullet} = \textcolor{magenta}{\bullet}$

$\{\textcolor{blue}{\bullet},\textcolor{magenta}{\bullet}\}^{\eta} = \textcolor{magenta}{\bullet}$

Recovering countable products from algebras

$w =$ 

To define $\pi(w)$ repeatedly simplify w by evaluating infixes

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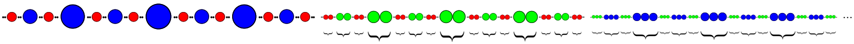
$w =$ 

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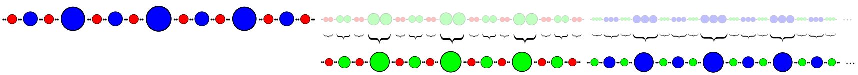
$w =$ 

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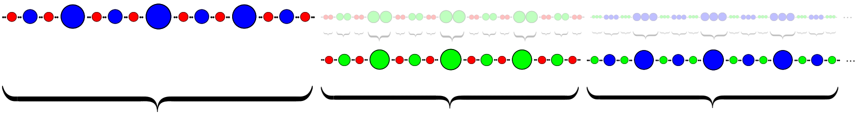

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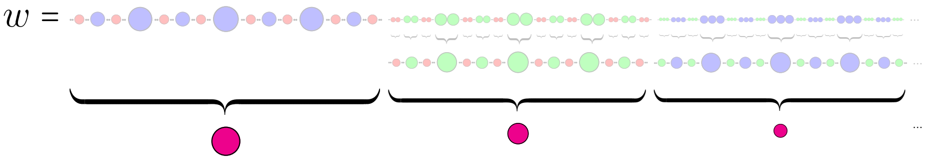

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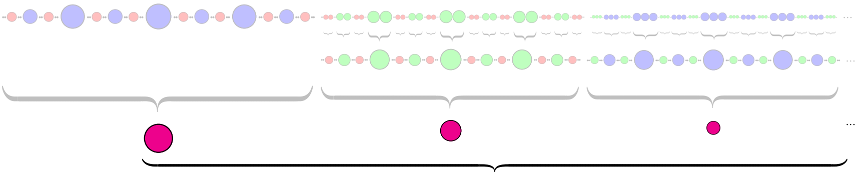

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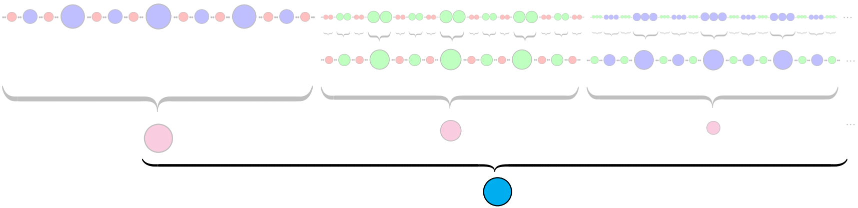
$w =$ 

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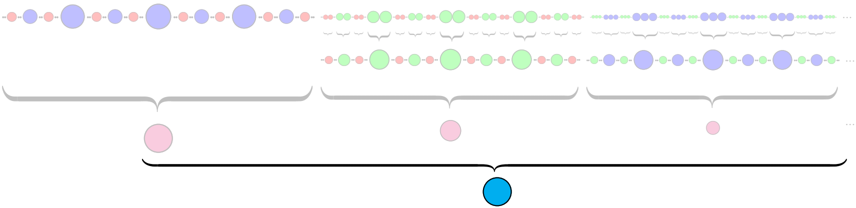

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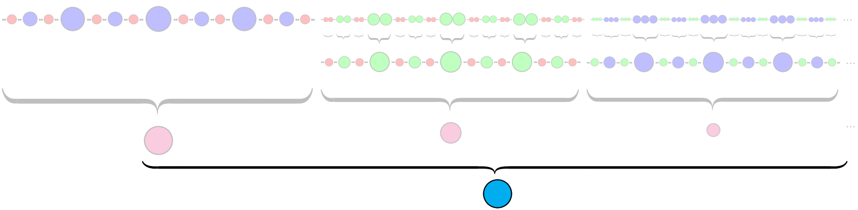

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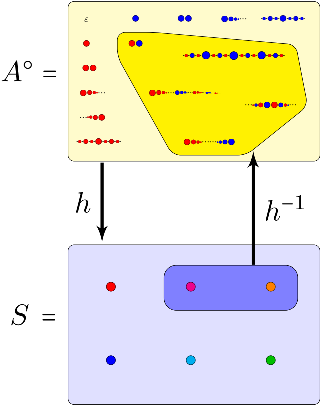
- existence: Theorems a-la Ramsey + Axiom of Choice
- well-definedness: Equations for associativity + Induction

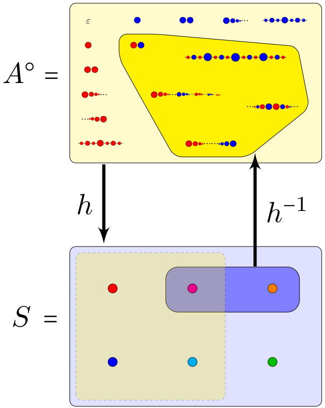
Deciding emptiness of recognizable languages



$$L = h^{-1}(F)$$

$$h(L) = h\big(h^{-1}(F)\big)$$

Deciding emptiness of recognizable languages



$$L = h^{-1}(F)$$

$$\begin{aligned} h(L) &= h\big(h^{-1}(F)\big) \\ &= F \cap h\big(A^\circ\big) \end{aligned}$$

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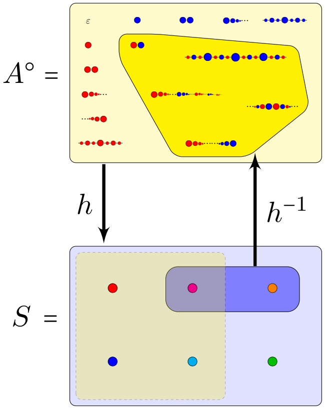

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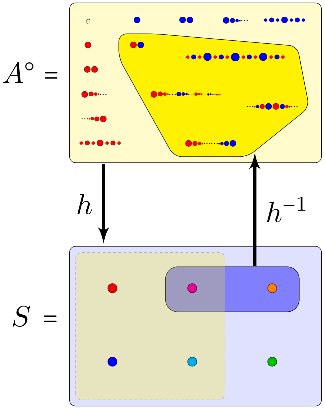

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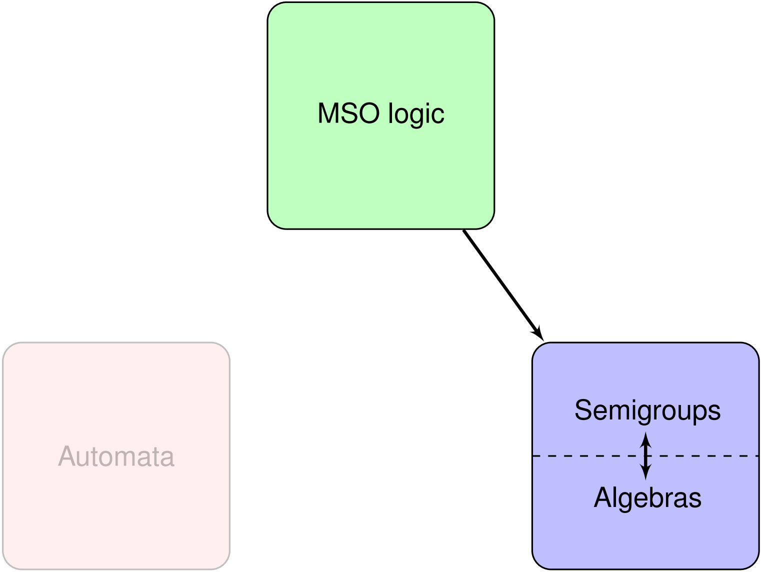
$L \neq \varnothing$ iff $F \cap \langle\, h(A) \,\rangle \neq \varnothing$

Translations between formalisms



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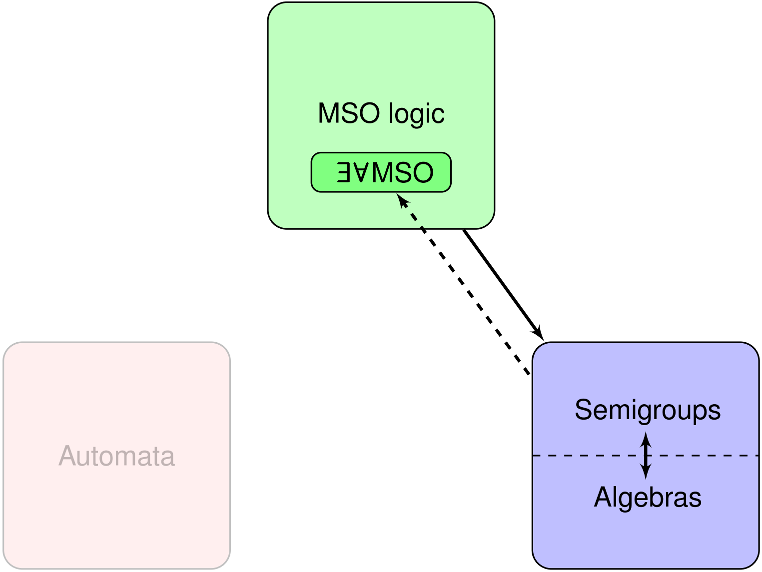

From semigroups to MSO in normal form

☞ Recognizable languages can be defined in **∃∀MSO**

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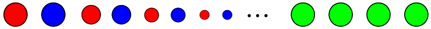
new evaluation strategy = Factorization Forest [Simon '90] [Colcombet '10]

= tree of small (bounded) height that eases evaluation of subwords via FO

Factorization forests

$w =$ 

Internal nodes of a factorization forest can have:

- **2 children** with arbitrary values
- **several children** with **same idempotent** $(e \cdot e = e)$

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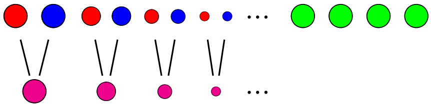
$w =$ 

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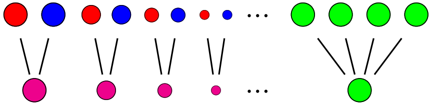

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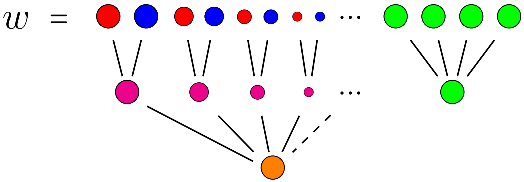

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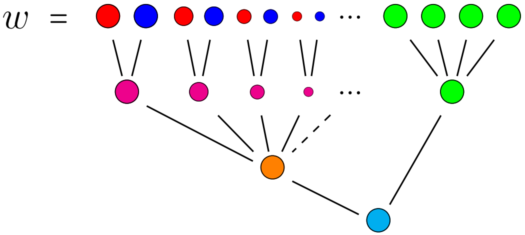

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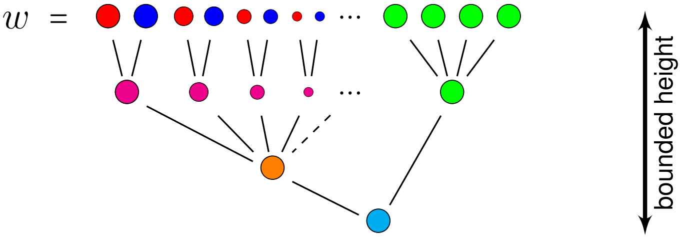

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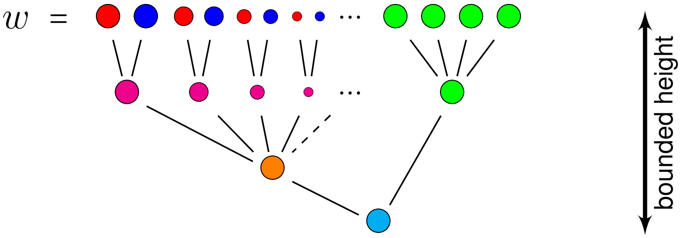

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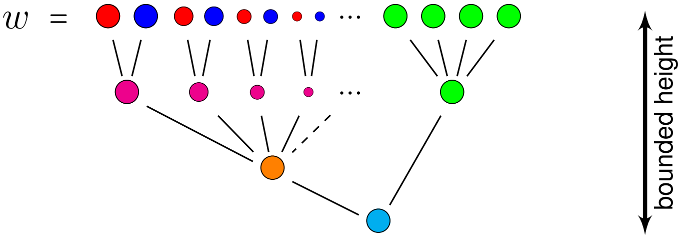

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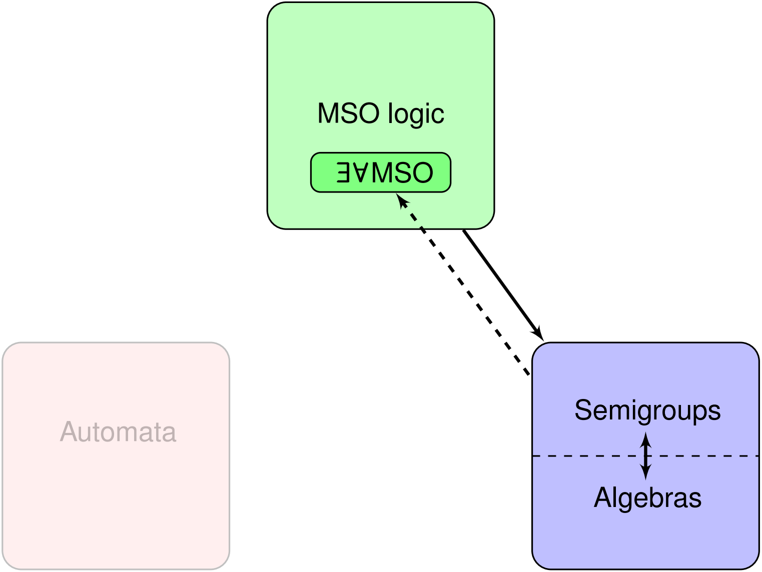
$$w \in L \iff \exists \text{ factorization forest } \bar{X}.\ \ \text{value}(w) \in F$$
$$\wedge\ \ \forall \text{ subword } Y.\ \forall \text{ factorization } Z.\ \text{value}(Y) = \prod_{Y_i \text{ factor of } Z} \text{value}(Y_i)$$

Translations between formalisms (cont'd)



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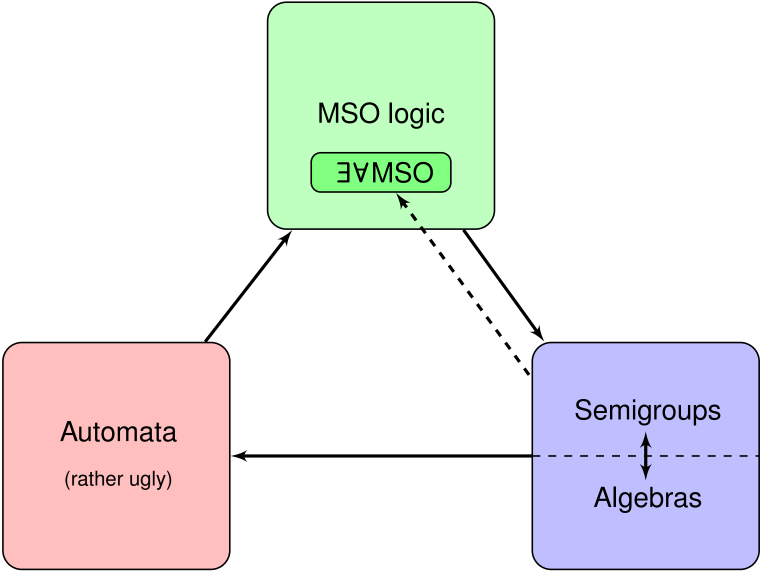

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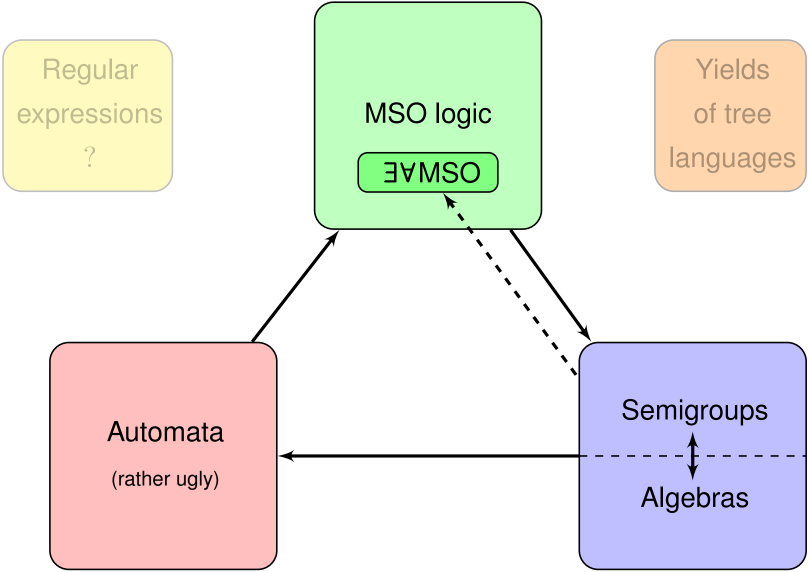
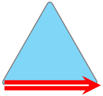


Other applications

- **Yields of trees**



L regular language of countable words

$$\Updownarrow$$

$T = \left\{ t \;:\; \text{yield}(t) \in L \right\}$ regular language of trees

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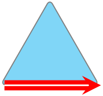

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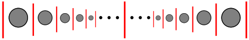

variables $x, X, \ldots$ for positions
$\hat{x}, \hat{X}, \ldots$ for cuts

MSO$[\mathbb{Q}, \hat{\mathbb{Q}}]$ is undecidable (like MSO$[\mathbb{R}]$)
MSO$[\mathbb{Q}, \hat{\mathbb{Q}}]$ defines same predicates over $\mathbb{Q}$ as MSO$[\mathbb{Q}]$

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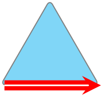

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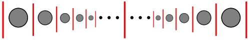

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- **Characterizations of FO-definable languages...**